Counterexamples to the Interpolating Property

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The following is a theorem proved by Zhao and Yan,

Theorem

For a binary delta-matroid $D = (E, F)$, the twist polynomial $\partial w_D(z)$ is either an even polynomial, an odd polynomial, or both even-interpolating and odd-interpolating.

The binary assumption here **is** necessary. Let's see when it fails.

Known Counterexample

Here is the counterexample in the nonbinary case provided by Yan:

Counterexample

$E = \{1, 2, \ldots, 8\}, \qquad D = (E, \mathcal{F}), \qquad \mathcal{F} = \{F \subseteq E \mid |F| \in \{0, 1, 3, 4\}\}$

It has the twist polynomial

$$\partial w_D(z) = 2z^4 + 72z^5 + 112z^7 + 70z^8.$$

We can see how this is obtained by considering the width of D when twisting with respect to a subset of cardinality k.

Understanding the Known Counterexample

We can consider all possible cardinalities of subsets, those being 0 through 8. Then think: what is the minimum cardinality feasible set when twisting by a subset A of cardinality k? If $k = 0$, the $r_{\min}(D\Delta A) = 0$. If $k = 1$, $r_{\min}(D\Delta A) = 0$. But if $k = 2$, $r_{\min}(D\Delta A) = 1$. We can see this by twisting the appropriate feasible set of cardinality 1 OR 3 to get a feasible set of cardinality 1.

In general, $r_{\min}(D\Delta A) = \min_{c \in \{0,1,3,4\}} |c - |A||$. On the other hand, we have that $r_{\max}(D\Delta A) = 8 - \min_{c \in \{0,1,3,4\}} |c - (8 - |A|)|$. The takeaway here is that the width here increases as the distance of the cardinality of A from the nearest element of our chosen set of cardinalities decreases, or as the same happens to the reflection of the cardinality of A across $\{0, 1, \ldots, 8\}$.

$$w(D\Delta A) = 8 - \min_{c \in \{0,1,3,4\}} |c - (8 - |A|)| - \min_{c \in \{0,1,3,4\}} |c - |A||$$

Image

Let A denote the set of edges we twist by (with $|A|$ shown by a black dot below the image and $8-|A|$ shown by a red dot), and consider the cardinality set (whose elements are shown by x in the image).

0	1	2	3	4	5	6	7	8
x	x		x	x				

When $n=0$, $w(D\Delta A) = 8 - 4 - 0 = 4$.

$$\partial w_D(z) = \cdot z^4 +$$

Image

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0	1	2	3	4	5	6	7	8
x	x		x	x				

When $n = 1$, $w(D\Delta A) = 8 - 3 - 0 = 5$.

$$\partial w_D(z) = \cdot z^4 + \cdot z^5 +$$

Image

Let A denote the set of edges we twist by (with $|A|$ shown by a black dot below the image and $8-|A|$ shown by a red dot), and consider the cardinality set (whose elements are shown by x in the image).

0	1	2	3	4	5	6	7	8
x	x		x	x				

When $n = 2$, $w(D\Delta A) = 8 - 2 - 1 = 5$.

$$\partial w_D(z) = \cdot z^4 + \cdot z^5 +$$

Image

Let A denote the set of edges we twist by (with $|A|$ shown by a black dot below the image and $8-|A|$ shown by a red dot), and consider the cardinality set (whose elements are shown by x in the image).

0	1	2	3	4	5	6	7	8
x	x		x	x				

When $n=3$, $w(D\Delta A) = 8 - 1 - 0 = 7$.

$$\partial w_D(z) = \cdot z^4 + \cdot z^5 + \cdot z^7 +$$

Image

Let A denote the set of edges we twist by (with $|A|$ shown by a black dot below the image and $8 - |A|$ shown by a red dot), and consider the cardinality set (whose elements are shown by x in the image).

0	1	2	3	4	5	6	7	8
x	x		x	x				

When $n = 4$, $w(D\Delta A) = 8 - {\color{red}0} - 0 = 8$.

$$\partial w_D(z) = \cdot z^4 + \cdot z^5 + \cdot z^7 + \cdot z^8$$

Other Counterexamples

Our analysis of this counterexample leads to a number of extra counterexamples for free. For example, the family of delta-matroids with $D_m = (E_m, \mathcal{F}_m)$ and

$$E_m = \{1, 2, \ldots, 8 + m\},\ \mathcal{F}_m = \{F \subseteq E \mid |F| \in \{0, 1, 3, 4, \ldots, 4 + m\}\}$$

where m is a nonnegative integer consists entirely of counterexamples to this interpolating property.

Conjectures

I conjecture that if a delta-matroid violates the interpolation property from the beginning of this presentation, it is the twist of some delta-matroid of the form

$$(\{1, 2, \ldots, n\}, \{F \subseteq \mathcal{P}(\{1, 2, \ldots, n\}) \mid |F| \in C\}),$$

where $C \subseteq \{0, 1, \ldots, n\}$ is the cardinality set. This is probably not true, but as far as I know there is no known counterexample not of this form, making it at least reasonable to try to disprove.

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Adam's conjecture is more reasonable, which is that any delta-matroid representable over any field satisfies the interpolating property from the beginning of the talk (already proved when the field is $\mathbb{F}_2$, since binary just means representable over $\mathbb{F}_2$). So, we would like to know if this counterexample, or any that it spawns, is representable over fields other than $\mathbb{F}_2$. Unfortunately, this is a hard question to answer.

References

G. Zhang, Q. Yan, Twist polynomial interpolation for binary delta-matroids, Preprint arXiv:2605.05601v1 [math.CO].